

Math 522 Exam 6 Solutions

1. Suppose that $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}$. Suppose that $a \equiv b \pmod{k}$, $|a| < \frac{k}{3}$, and $|b| \leq \frac{2k}{3}$. Prove that $a = b$.

Because $a \equiv b$, there is some $n \in \mathbb{Z}$ with $a - b = nk$. For convenience, we rewrite our inequalities as $-\frac{k}{3} < a < \frac{k}{3}$ and $-\frac{2k}{3} \leq b \leq \frac{2k}{3}$.

Because $a < \frac{k}{3}$ and $-b \leq \frac{2k}{3}$, their sum $a - b < \frac{k}{3} + \frac{2k}{3} = k$. Hence $nk < k$ so $n < 1$. Because $-a < \frac{k}{3}$ and $b \leq \frac{2k}{3}$, their sum $b - a < \frac{k}{3} + \frac{2k}{3} = k$, so $a - b > -k$. Hence $nk > -k$ so $n > -1$. But the only integer n satisfying $-1 < n < 1$ is $n = 0$, so $a = b$.

2. Determine whether there is a complete residue system modulo 5 consisting entirely of even integers. Be sure to fully justify your answer.

The answer is yes, and it suffices to exhibit one example. We start with the standard complete residue system $\{0, 1, 2, 3, 4\}$. Three of these are even already, and $1 + 5 = 6, 3 + 5 = 8$, so we may equally have $\{0, 6, 2, 8, 4\}$ as the desired solution. These are pairwise incongruent because we built this from an existing c.r.s.; if we had built this from scratch we'd need to justify that no two elements are congruent.

Note: This is possible because $\gcd(2, 5) = 1$. Otherwise, impossible.